REPORT DOCUMENTATION PAGE

Form Approved OMB NO. 0704-0188

1. REPORT DATE (DD-MM-YYYY)	2. REPORT TYPE	3. DATES COVERED (From - To)
08-08-2011	Conference Proceeding	-

4. TITLE AND SUBTITLE	5a. CONTRACT NUMBER
Equivalent Standard Deviation to Convert High-reliability Model to Low-reliability Model for Efficiency of Sampling-based RBDO	W911NF-09-1-0250
	5b. GRANT NUMBER
	5c. PROGRAM ELEMENT NUMBER 611102
6. AUTHORS	5d. PROJECT NUMBER
Ikjin Lee, Kyung K. Choi, David Gorsich	5e. TASK NUMBER
	5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAMES AND ADDRESSES	8. PERFORMING ORGANIZATION REPORT NUMBER
University of Iowa @ Iowa City Office of Sponsored Programs The University of Iowa Iowa City, IA 52242 -	

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)	10. SPONSOR/MONITOR'S ACRONYM(S)
U.S. Army Research Office P.O. Box 12211 Research Triangle Park, NC 27709-2211	ARO
	11. SPONSOR/MONITOR'S REPORT NUMBER(S) 56025-NS.29

12. DISTRIBUTION AVAILIBILITY STATEMENT

Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES

The views, opinions and/or findings contained in this report are those of the author(s) and should not contrued as an official Department of the Army position, policy or decision, unless so designated by other documentation.

14. ABSTRACT

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15. SUBJECT TERMS

Very Small Probability of Failure, Sampling-based RBDO,Monte Carlo Simulation, Score Function, Copula, SurrogateModel.

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	15. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
a. REPORT	b. ABSTRACT	c. THIS PAGE			Kyung Choi
UU	UU	UU	UU		19b. TELEPHONE NUMBER 319-335-5684

Standard Form 298 (Rev 8/98)
Prescribed by ANSI Std. Z39.18

Report Title

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Conference Name: ASME 2011 International Design Engineering Technical Conferences & Computers and Information in En

Conference Date: August 28, 2011

Proceedings of the ASME 2011 International Design Engineering Technical Conferences &
Computers and Information in Engineering Conference
IDETC/CIE 2011
August 28 − 31, 2011, Washington, D.C., USA

DETC2011-47537

DRAFT

Equivalent Standard Deviation to Convert High-reliability Model to Low-reliability Model for Efficiency of Sampling-based RBDO

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KEYWORDS

Very Small Probability of Failure, Sampling-based RBDO, Monte Carlo Simulation, Score Function, Copula, Surrogate Model.

1. INTRODUCTION

Surrogate models or meta-models have been widely used for reliability-based design optimization (RBDO) of various engineering applications when accurate sensitivities of performance functions are not available [1-6]. When surrogate models are used for RBDO, sampling-based reliability analysis methods to evaluate probabilistic constraints of RBDO are often adapted due to their computational simplicity. The most straightforward approach among sampling techniques is the direct Monte Carlo simulation (MCS) [7]. Using the MCS, the probability of failure can be estimated by counting the number of samples within the failure region and dividing it by the total number of samples. The main concern for the MCS is the computational cost because it is well known that the total number of samples required to obtain a reasonably accurate estimate is proportional to the inverse of the probability of failure [8], which implies that a very large

number of samples will be required for the MCS if the target probability of failure is very small, for example, 4~6σ design.

To enhance the computational efficiency of the MCS, Latin Hypercube sampling (LHS) and its modification [9-13] can be used for the reliability analysis. The LHS is known to be more efficient than the MCS since its stratification properties allow for the estimation of the probability of failure with a relatively small sample size [12], and it is also shown that the LHS can save more than 50% of the computational cost of the direct MCS [13]. To further enhance the computational efficiency of the sampling scheme, variations of the MCS, including the importance sampling [13-15], subset simulation [16], and directional sampling [17,18], have been proposed. All these improvements of the direct MCS attempt to reduce the number of samples by either allocating samples more effectively on the sampling domain or moving the sampling domain near the limit state function where performance functions have zero values. However, even if the number of samples is reduced by using the efficient sampling schemes, a large number of samples are still required for a very small target probability of failure. Furthermore, if the sampling domain moves to the vicinity of the limit state function as in the importance sampling, as many surrogate models as the number of RBDO constraints have to be generated at a given design when the local window concept is used for the sampling-based RBDO to improve its accuracy. In addition, it requires additional computations to move the sampling domain to the vicinity of the limit state function, which makes the sampling-based RBDO further inefficient. To avoid this, a global window concept can be used to generate the surrogate models; however, it causes accuracy problems, especially for high-dimensional problems.

The main objective of this paper is to propose a methodology to convert an RBDO problem with a very small target probability of failure to an RBDO problem with a relatively high probability of failure by increasing the input standard deviations to reduce the computational cost of the direct MCS for the sampling-based RBDO. For this, the exact relationship between the probability of failure and input standard deviations is derived for linear performance functions with independent normal random inputs, and then the relationship is generalized for any random input and performance functions. To derive the general relationship between the probability of failure and input standard deviation, the first-order score function for the input standard deviations is introduced [19-21]. After finding the relationship, a concept of an equivalent standard deviation, which is an increased standard deviation corresponding to the model with high probability of failure, is also proposed to be used for sampling-based RBDO problems. Since the proposed method is applied to the sampling-based RBDO, accurate surrogate models are naturally used for the reliability analysis instead of computationally expensive computer simulations. For the generation of accurate surrogate models, the Dynamic Kriging method [22] can be utilized. Even if it is developed to be used with surrogate models, the proposed method is applicable to the sampling-based RBDO using actual computer simulations if the model is not too computationally demanding since the proposed method can reduce the number of samples significantly, especially when used in conjunction with the efficient sampling schemes.

The paper is organized as follows. Section 2 briefly reviews the formulation of the sampling-based RBDO and the stochastic sensitivity analysis since it helps understand the proposed method. Section 3 shows how to derive the relationship between the probability of failure and input standard deviations for general random inputs and performance functions. Section 4 explains the concept of the equivalent standard deviation for the sampling-based RBDO. Section 5 illustrates with numerical examples the efficiency and accuracy of the proposed method compared with results obtained using the original random input. Finally, this paper is concluded in Section 6.

2. SAMPLING-BASED RBDO

In this section, we briefly review the concept of the sampling-based RBDO, which will help us understand the proposed method in Section 3. Section 2.1 explains the formulation of the sampling-based RBDO and the evaluation of probabilistic constraints. Section 2.2 reviews the stochastic sensitivity analysis using the score function for mean values. Finally, Section 2.3 shows accuracy of sampling techniques and justification of the proposed method, which is explained in detail in Section 3.

2.1 Formulation

The mathematical formulation of a general RBDO problem is expressed as

$$\begin{aligned} &\text{minimize} \quad \mathrm{Cost}(\mathbf{d}) \\ &\text{subject to} \quad P[G_j(\mathbf{X})>0] \le P_{F_j}^{\mathrm{Tar}}, \quad j=1,\cdots,NC \\ &\qquad\qquad \mathbf{d}^{\mathrm{L}} \le \mathbf{d} \le \mathbf{d}^{\mathrm{U}}, \quad \mathbf{d}\in \mathrm{R}^{ndv} \text{ and } \mathbf{X}\in \mathrm{R}^{nrv} \end{aligned} \tag{1}$$

where $\mathbf{d}=\{d_i\}^{\mathrm{T}}=\boldsymbol{\mu}(\mathbf{X})$ is the design vector, which is the mean value of the N-dimensional random vector $\mathbf{X}=\{X_1, X_2, \cdots, X_N\}^{\mathrm{T}}$; $P[\bullet]$ represents the probability measure; $P_{F_j}^{\mathrm{Tar}}$ is the target probability of failure for the j^{th} constraint; and NC, ndv, and nrv are the number of probabilistic constraints, design variables, and random variables, respectively.

A reliability analysis involves calculation of the probability of failure, denoted by P_F and shown in Eq. (1) as $P[G_j(\mathbf{X})>0]$, which is defined using a multi-dimensional integral

$$P_F(\boldsymbol{\psi}) \equiv P[\mathbf{X}\in\Omega_F] = \int_{\mathbb{R}^N} I_{\Omega_F}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\psi})d\mathbf{x} = E\left[I_{\Omega_F}(\mathbf{X})\right] \tag{2}$$

where $\boldsymbol{\psi}$ is a vector of distribution parameters, which usually includes the mean ($\boldsymbol{\mu}$) and/or standard deviation ($\boldsymbol{\sigma}$) of the random input $\mathbf{X}$; Ω_F is the failure set; $f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\psi})$ is a joint input probability density function (PDF) of $\mathbf{X}$; and $E[\bullet]$ represents the expectation operator. The failure set is defined as $\Omega_F \equiv \{\mathbf{x}: G_j(\mathbf{x})>0\}$ for component reliability analysis of the j^{th} constraint function $G_j(\mathbf{x})$, and $\Omega_F \equiv \left\{\mathbf{x}: \bigcup_{j=1}^{NC} G_j(\mathbf{x})>0\right\}$

and $\Omega_F \equiv \left\{ \mathbf{x} : \bigcap_{j=1}^{NC} G_j(\mathbf{x}) > 0 \right\}$ for series system and parallel system reliability analysis of *NC* performance functions, respectively [20,23]. $I_{\Omega_F}(\mathbf{x})$ in Eq. (2) is called an indicator function and defined as

$$I_{\Omega_F}(\mathbf{x}) \equiv \begin{cases} 1, & \mathbf{x} \in \Omega_F \\ 0, & otherwise \end{cases} \tag{3}$$

To carry out RBDO in Eq. (1) using gradient-based optimization methods, it is required to know the function value and its sensitivities of the probabilistic constraints at a given design point. However, in most engineering applications, it is difficult to obtain accurate sensitivities. For engineering applications where accurate sensitivities are not available, surrogate models have been widely used to carry out design optimization. Once an accurate surrogate model is available for the design optimizations, sampling techniques such as the direct MCS or more efficient LHS can be applied to estimate the probability of failure with an affordable computational cost.

Denote the surrogate model for a constraint function $G_j(\mathbf{X})$ as $\hat{G}_j(\mathbf{X})$. Then, by applying the MCS or LHS to the surrogate model, the probabilistic constraints in Eq. (1) can be approximated as

$$P_{F_j} \equiv P[G_j(\mathbf{X}) > 0] \cong \frac{1}{K} \sum_{k=1}^{K} I_{\hat{\Omega}_F}(\mathbf{x}^{(k)}) \le P_{F_j}^{\mathrm{Tar}} \tag{4}$$

where K is the sample size, $\mathbf{x}^{(k)}$ is the k^{th} realization of $\mathbf{X}$, and the failure set $\hat{\Omega}_F$ for the surrogate model is defined as $\hat{\Omega}_F \equiv \left\{ \mathbf{x} : \hat{G}_j(\mathbf{x}) > 0 \right\}$.

2.2 Stochastic Sensitivity Analysis

In addition to the probability of failure shown in Eq. (2), its sensitivity with respect to a design variable μ_i is required to carry out RBDO in Eq. (1). Taking the partial derivative of Eq. (2) with respect to μ_i yields

$$\frac{\partial P_F(\boldsymbol{\mu})}{\partial \mu_i} = \frac{\partial}{\partial \mu_i} \int_{\mathbb{R}^N} I_{\Omega_F}(\mathbf{x}) f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu}) d\mathbf{x} . \tag{5}$$

In Eq. (5), distribution parameter $\boldsymbol{\psi}$ become $\boldsymbol{\mu}$ because $\boldsymbol{\mu}$ is the design vector and $\boldsymbol{\sigma}$ is assumed to be independent of $\boldsymbol{\mu}$. Since the differential and integral operators can be interchanged if the integrand in Eq. (5) is bounded due to the Lebesgue dominated convergence theorem [19,20], Eq. (5) can be rewritten as

$$\begin{aligned} \frac{\partial P_F(\boldsymbol{\mu})}{\partial \mu_i} &= \int_{\mathbb{R}^N} I_{\Omega_F}(\mathbf{x}) \frac{\partial f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial \mu_i} d\mathbf{x} \\ &= \int_{\mathbb{R}^N} I_{\Omega_F}(\mathbf{x}) \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial \mu_i} f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu}) d\mathbf{x} . \\ &= E\left[I_{\Omega_F}(\mathbf{x}) \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial \mu_i} \right] \end{aligned} \tag{6}$$

The partial derivative of the log function of the joint PDF in Eq. (6) with respect to μ_i is known as the first-order score function for μ_i and is denoted as

$$s_{\mu_i}^{(1)}(\mathbf{x};\boldsymbol{\mu}) \equiv \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\mu})}{\partial \mu_i} . \tag{7}$$

The first-order score functions for specific marginal and joint distribution types are listed in Ref. [21] in detail.

In a similar manner to Eq. (4) for the probability of failure calculation, its sensitivity is obtained using the first-order score function in Eq. (7) and applying sampling techniques to Eq. (6) as

$$\frac{\partial P_{F_j}}{\partial \mu_i} \cong \frac{1}{K} \sum_{k=1}^{K} I_{\hat{\Omega}_F}(\mathbf{x}^{(k)}) s_{\mu_i}^{(1)}(\mathbf{x}^{(k)};\boldsymbol{\mu}) . \tag{8}$$

2.3 Accuracy of Simulation for Reliability Analysis

The percentage error of the MCS to compute the probability of failure by Eq. (4) can be measured using the 95% confidence interval of the estimated probability of failure and given by [24]

$$\varepsilon_{MCS} = \sqrt{\frac{(1 - P_F^{\mathrm{Tar}})}{K \times P_F^{\mathrm{Tar}}}} \times 200\% . \tag{9}$$

To estimate the percentage error of Eq. (8), Eq. (8) can be rewritten as

$$\begin{aligned} \frac{\partial P_{F_j}}{\partial \mu_i} &= \frac{1}{K} \sum_{k_f=1}^{K_f} I_{\hat{\Omega}_{F_j}}(\mathbf{x}^{(k_f)}) s_{\mu_i}^{(1)}(\mathbf{x}^{(k_f)};\boldsymbol{\mu}) \\ &= \frac{1}{K} \sum_{k_f=1}^{K_f} s_{\mu_i}^{(1)}(\mathbf{x}^{(k_f)};\boldsymbol{\mu}) = \frac{K_f}{K} \mu_{s_f} = P_{F_j} \mu_{s_f} \end{aligned} \tag{10}$$

where K_f is the number of failed samples and μ_{s_f} is the mean value of the score function values for the failed samples. Hence, the percentage error for the MCS to compute Eq. (8) can be measured by $\varepsilon_{MCS} \mu_{s_f}$.

To see the effect of the percentage error in Eq. (9) on the target probability of failure, consider an example where the MCS sample size K is 500,000 and $P_F^{\mathrm{Tar}} = 2.275\%$. Then, ε_{MCS} is 1.85% of the target probability of failure, which means that there exists 95% probability that the probability of failure estimated using the MCS will be between 2.233% and 2.317% (i.e., ±1.85% interval of 2.275%) with 500,000

samples. If the target probability of failure is very small, for example, $P_F^{\text{Tar}} = 0.003167\%$, which is called a 4σ design, then, 369,024,089 MCS samples are required to satisfy the same percentage error of 1.85%. Or, if the MCS sample size K is 500,000, then the percentage error becomes 50.26%, which is too large to be used for RBDO. Even if the surrogate model is not computationally demanding and many model evaluations can be performed, too many samples can cause computer memory problems and make the sampling-based RBDO extremely slow, especially when implicit surrogate models such as the Kriging model are used.

To obtain more accurate results, the LHS can be utilized, which is known to be more accurate for the probability of failure calculation than the MCS when the same number of samples is used [9-13]. However, even with the LHS, the sample size should be very large to be accurate for very small probability of failure problems. Hence, in this paper, the MCS is used for the sampling scheme since it does not change the main point of the paper.

As mentioned above, for a very small target probability of failure such as 4~6σ design, the probability of failure and its sensitivities computed using Eqs. (4) and (8), respectively, could be inaccurate unless a sufficient number of samples are used, which may not be possible due to high computational cost. Thus, the sampling-based RBDO for a high-reliability model could yield a wrong optimum design due to inaccurate estimation of the probability of failure. To overcome this, it is necessary to convert the high-reliability model to a lower-reliability model by increasing input standard deviations and finding an equivalent probability of failure corresponding to the increased standard deviations, which is the main purpose of the paper and will be explained in the subsequent section.

3. EQUIVALENT STANDARD DEVIATION FOR RELIABILITY ANALYSIS

3.1 For Independent Normal Random Variables and Linear Limit State Function

Consider a linear performance function as

$$G(\mathbf{X}) = \sum_{i=1}^{N} a_i X_i + a_0 = \mathbf{a}^{\mathrm{T}}\mathbf{X} + a_0 \tag{11}$$

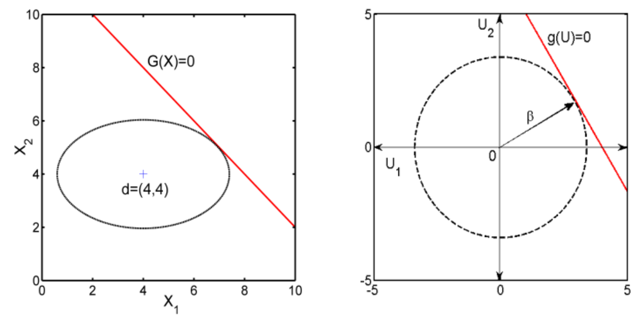
where X_i have a normal distribution as $X_i \sim N(\mu_i, \sigma_i^2)$. Using the Rosenblatt transformation [25] from the X-space to the U-space, X_i can be expressed as

$$X_i = \sigma_i U_i + \mu_i, \tag{12}$$

and by inserting Eq. (12) into Eq. (11), the performance function can be rewritten in the U-space as [26]

$$\begin{aligned} g(\mathbf{U}) &= \sum_{i=1}^{N} a_i(\sigma_i U_i + \mu_i) + a_0 \\ &= \sum_{i=1}^{N} a_i \sigma_i U_i + \sum_{i=1}^{N} a_i \mu_i + a_0 = \mathbf{a}_1^{\mathrm{T}}\mathbf{U} + b_0 \end{aligned} \tag{13}$$

where $\mathbf{a}_1^{\mathrm{T}} = \{a_i\sigma_i\}^{\mathrm{T}}$ and $b_0 = \sum_{i=1}^{N} a_i\mu_i + a_0 = \mathbf{a}^{\mathrm{T}}\boldsymbol{\mu} + a_0 = G(\boldsymbol{\mu})$. A 2-D example for linear performance functions in the X-space and the U-space is illustrated in Fig. 1(a) and (b), respectively.



(a) X-space (b) U-space

Figure 1. Linear Performance Function

The reliability index β is defined as the minimum distance from the origin in the U-space to the limit state function [27], which is defined as $g(\mathbf{U}) = 0$, as shown in Fig. 1(b) and expressed as

$$\beta = -\frac{b_0}{\|\mathbf{a}_1\|}. \tag{14}$$

Since failure is defined if $G(\mathbf{X}) > 0$ and $b_0 = G(\boldsymbol{\mu})$, the reliability index β is positive when a function value at the design point is negative, which implies the design point is located in the feasible region. Accordingly, the reliability index becomes negative if the design point is located in the infeasible domain.

For a linear limit state function in the U-space, the probability of failure is analytically given using the reliability index by

$$P_F = \Phi(-\beta) = \Phi\left(\frac{b_0}{\|\mathbf{a}_1\|}\right), \tag{15}$$

and since $\mathbf{a}_1^{\mathrm{T}} = \{a_i\sigma_i\}^{\mathrm{T}}$, the probability of failure in Eq. (15) can be expressed in terms of input standard deviations as

$$P_F(\boldsymbol{\sigma}) = \Phi\left(\frac{b_0}{\sqrt{\sum_{i=1}^{N}(a_i\sigma_i)^2}}\right). \tag{16}$$

The input standard deviation (σ_i) of the i^{th} random variable can be expressed as $\sigma_i = \delta_i \sigma_i^o$ using the ratio (δ_i) where σ_i^o is the current standard deviation for the i^{th} random variable. Then, the probability of failure at a given design $\mathbf{d} = \boldsymbol{\mu}(\mathbf{X})$ for a linear limit state function in the U-space is expressed as a function of the ratio $\boldsymbol{\delta}$ as

$$P_F(\boldsymbol{\delta}) = \Phi\left(\frac{b_0}{\sqrt{\sum_{i=1}^{N}(\delta_i a_i \sigma_i^o)^2}}\right). \tag{17}$$

Hence, for a linear limit state function in the U-space, the probability of failure is exactly computed according to the ratio of the input standard deviation ($\boldsymbol{\delta}$). Inversely, we can find an exact $\boldsymbol{\delta}$ value corresponding to the target probability of failure at the current design, which will be explained in detail in Section 4.1.

3.2 For General Random Inputs and Performance Functions

Even if it is exact for linear performance functions with independent normal random inputs, Eq. (17) cannot be directly used for general cases where random inputs could be non-normal and/or correlated, or performance functions are nonlinear. Therefore, to generalize Eq. (17) so that it can be used for any random inputs, either correlated or independent, and nonlinear performance functions, Eq. (17) is modified as

$$P_F(\boldsymbol{\delta}) = c_0 \Phi\left(-\frac{1}{\sqrt{\sum_{i=1}^{N}(c_i \delta_i \sigma_i^o)^2}}\right) \tag{18}$$

where c_0 and c_i need to be determined using the current probability of failure where $\boldsymbol{\delta} = \mathbf{1}$

$$P_F^{\text{cur}} = c_0 \Phi\left(-\frac{1}{\sqrt{\sum_{i=1}^{N}(c_i \sigma_i^o)^2}}\right) = c_0 \Phi(d_0) \tag{19}$$

and its sensitivity with respect to δ_j

$$\begin{aligned} \left.\frac{\partial P_F}{\partial \delta_j}\right|_{\boldsymbol{\delta}=\mathbf{1}} &= -c_0 \phi(d_0) \times d_0^3 \times (c_j \sigma_j^o)^2 \\ &= -\frac{P_F^{\text{cur}}}{\Phi(d_0)} \phi(d_0) \times d_0^3 \times (c_j \sigma_j^o)^2 \end{aligned}. \tag{20}$$

Thus, summing up Eq. (20) from j=1 to j=N yields

$$\begin{aligned} \sum_{j=1}^{N} \left.\frac{\partial P_F}{\partial \delta_j}\right|_{\boldsymbol{\delta}=\mathbf{1}} &= -\frac{P_F^{\text{cur}}}{\Phi(d_0)} \phi(d_0) \times d_0^3 \times \sum_{j=1}^{N}(c_j \sigma_j^o)^2 \\ &= -\frac{P_F^{\text{cur}}}{\Phi(d_0)} \phi(d_0) \times d_0 \end{aligned} \tag{21}$$

from Eq. (19). Once the left-hand side of Eq. (21) is known, Eq. (21) can be easily solved for d_0 using numerical root finding methods such as "fzero" in Matlab, which does not require extra function calls of the original performance functions and thus is computationally trivial. After obtaining d_0, c_j can be obtained using Eq. (20). Then, the probability of failure in Eq. (18) can be expressed using d_0 and c_j as

$$P_F(\boldsymbol{\delta}) = \frac{P_F^{\text{cur}}}{\Phi(d_0)} \Phi\left(-\frac{1}{\sqrt{\sum_{i=1}^{N}(c_i \delta_i \sigma_i^o)^2}}\right). \tag{22}$$

The sensitivity of the probability of failure with respect to δ_i in Eq. (21) is given using the chain rule as

$$\left.\frac{\partial P_F}{\partial \delta_i}\right|_{\boldsymbol{\delta}=\mathbf{1}} = \sigma_i^o \left.\frac{\partial P_F}{\partial \sigma_i}\right|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^o}, \tag{23}$$

and the sensitivity of the probability of failure with respect to σ_i is expressed in a similar manner to Eq. (6) as

$$\begin{aligned} \left.\frac{\partial P_F(\boldsymbol{\sigma})}{\partial \sigma_i}\right|_{\boldsymbol{\sigma}=\boldsymbol{\sigma}^o} &= \int_{\mathbb{R}^N} I_{\Omega_F}(\mathbf{x}) \frac{\partial f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\sigma})}{\partial \sigma_i} d\mathbf{x} \\ &= E\left[I_{\Omega_F}(\mathbf{x}) \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\sigma})}{\partial \sigma_i}\right] \end{aligned}. \tag{24}$$

The partial derivative of the log function of the joint PDF with respect to σ_i is known as the first-order score function for σ_i and is denoted as [19,20]

$$s_{\sigma_i}^{(1)}(\mathbf{x};\boldsymbol{\sigma}) \equiv \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\sigma})}{\partial \sigma_i}. \tag{25}$$

To further derive the first-order score function for σ_i, first consider statistically independent N−dimensional random input $\mathbf{X}$. Then, the joint PDF of $\mathbf{X}$ is expressed as multiplication of its marginal PDFs as

$$f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\sigma}) = \prod_{i=1}^{N} f_{X_i}(x_i;\sigma_i) \tag{26}$$

where $f_{X_i}(x_i;\sigma_i)$ is the marginal PDF corresponding to the i^{th} random variable X_i. Therefore, for statistically independent random variables, the first-order score function for σ_i is expressed as

$$s_{\sigma_i}^{(1)}(\mathbf{x};\boldsymbol{\sigma}) \equiv \frac{\partial \ln f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\sigma})}{\partial \sigma_i} = \frac{\partial \ln f_{X_i}(x_i;\sigma_i)}{\partial \sigma_i}. \tag{27}$$

Since the marginal PDF is analytically available as listed in Ref. 21, the derivation of Eq. (27) for any distribution type is very straightforward. The first-order score functions for σ_i for the normal, lognormal, and Gumbel distributions are listed in Table 1 where distribution parameters for each PDF are also explained in Ref. 21.

Table 1. First-Order Score Function for σ_i for Independent Random Variables

Marginal PDF	First-Order Score Function, $s_{\sigma_i}^{(1)}(\mathbf{x};\boldsymbol{\sigma})$
Normal	$\frac{1}{\sigma_i}\left[\left(\frac{x_i-\mu_i}{\sigma_i}\right)^2-1\right]$
Log-normal	$-\frac{1}{\bar{\sigma}_i}\frac{\partial\bar{\sigma}_i}{\partial\sigma_i}+\frac{1}{\bar{\sigma}_i^2}\left(\frac{\ln x_i-\bar{\mu}_i}{\bar{\sigma}_i}\right)\times\left[\bar{\sigma}_i\frac{\partial\bar{\mu}_i}{\partial\sigma_i}+(\ln x_i-\bar{\mu}_i)\frac{\partial\bar{\sigma}_i}{\partial\sigma_i}\right]$
Gumbel	$\frac{1}{\sigma_i}\left[\alpha_i\left(x_i-\mu_i\right)\left(1-e^{-\alpha_i(x_i-v_i)}\right)-1\right]$

For a bivariate correlated random input $\mathbf{X}=\left\{X_i,X_j\right\}^{\mathrm{T}}$, the joint PDF of $\mathbf{X}$ is expressed as [27-29]

$$f_{\mathbf{X}}(\mathbf{x};\boldsymbol{\sigma})=c(u,v;\theta)f_{X_i}(x_i;\sigma_i)f_{X_j}(x_j;\sigma_j) \tag{28}$$

where c is called a copula function and defined as

$$c(u,v;\theta)\equiv\frac{\partial^2C(u,v;\theta)}{\partial u\partial v}=C_{,uv}(u,v;\theta) \tag{29}$$

and $u=F_{X_i}(x_i;\sigma_i)$ and $v=F_{X_j}(x_j;\sigma_j)$ are CDFs for X_i and X_j, respectively; θ is the correlation coefficient between X_i and X_j; and C is a copula function [28,30]. Commonly used copula functions and their density functions are listed in Refs. 21 and 30.

Accordingly, using Eq. (28), the first-order score function in Eq. (25) for the correlated bivariate input is expressed as

$$s_{\sigma_i}^{(1)}(\mathbf{x};\boldsymbol{\mu})=\frac{\partial\ln c(u,v;\theta)}{\partial\sigma_i}+\frac{\partial\ln f_{X_i}(x_i;\sigma_i)}{\partial\sigma_i}. \tag{30}$$

The derivation of the first term of the right-hand side of Eq. (30) is also straightforward from analytic forms of copula density functions and listed in Table 2 for the Clayton, Frank, Gaussian, and independent copula where $\Phi(\bullet)$ and $\phi(\bullet)$ are the standard normal CDF and PDF, respectively, given by

$$\Phi(u)=\int_{-\infty}^{u}\phi(\xi)d\xi=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u}\exp\left(-\frac{1}{2}\xi^2\right)d\xi, \tag{31}$$

and the second term of the right-hand side of Eq. (30) is identical to Eq. (27), so it can be obtained from Table 1. One can see from Table 2 that the log-derivative of copula density function with respect to σ_i is identical to the log-derivative of copula density function with respect to μ_i shown in Table 5 of Ref. 21 except the term $\frac{\partial u}{\partial\sigma_i}$, which is a partial derivative of a marginal CDF with respect to σ_i and listed in Table 3. One can also see from Table 2 that Eq. (30) is identical to Eq. (27) if the independent copula is used, which means that the independent random input is a special case of the correlated random input where the independent copula is used.

Table 2. Log-derivative of Copula Density Function

Copula Type	$\frac{\partial\ln c(u,v;\theta)}{\partial\sigma_i}$
Clayton	$\left(-\frac{1+\theta}{u}+\frac{(2\theta+1)u^{-(1+\theta)}}{u^{-\theta}+v^{-\theta}-1}\right)\frac{\partial u}{\partial\sigma_i}$
Frank	$\theta\left[\frac{2\left(e^{\theta(1+u)}-e^{\theta(u+v)}\right)}{e^{\theta}-e^{\theta(1+u)}-e^{\theta(1+v)}+e^{\theta(u+v)}}+1\right]\frac{\partial u}{\partial\sigma_i}$
Gaussian	$\left[\frac{\Phi^{-1}(u)}{\phi(\Phi^{-1}(u))}+\frac{\theta\Phi^{-1}(v)-\Phi^{-1}(u)}{\phi(\Phi^{-1}(u))(1-\theta^2)}\right]\frac{\partial u}{\partial\sigma_i}$
Independent	0

Table 3. Partial Derivative of Marginal Distribution

Marginal PDF	$\frac{\partial u}{\partial\sigma_i}$
Normal	$-\frac{x-\mu_i}{\sigma_i^2}\phi\left(\frac{x-\mu_i}{\sigma_i}\right)$
Log-normal	$-\frac{1}{\bar{\sigma}_i^2}\left(\frac{\partial\bar{\mu}_i}{\partial\sigma_i}\bar{\sigma}_i+\frac{\partial\bar{\sigma}_i}{\partial\sigma_i}(\ln x-\bar{\mu}_i)\right)\phi\left(\frac{\ln x-\bar{\mu}_i}{\bar{\sigma}_i}\right)$
Gumbel	$-\frac{u}{\sigma_i}\left[1+\alpha_i\left(x_i-\mu_i\right)e^{-\alpha_i(x_i-v_i)}\right]$

4. EQUIVALENT STANDARD DEVIATION FOR SAMPLING-BASED RBDO

To apply Eq. (22) to the sampling- based RBDO, the new target probability of failure which is denoted as $P_{F_{new}}^{\mathrm{Tar}}$ needs to be set up. Then, the objective of this section is to find the equivalent standard deviation which is defined as the increased standard deviation to satisfy the new target probability of failure at the optimum design. However, Eq. (22) cannot be directly applied to the sampling-based RBDO since the current probability of failure changes during the design iteration. Furthermore, if the sampling scheme is utilized to estimate the current probability of failure and find the equivalent standard deviation at a given design, it is a significant computational effort. Thus, it is necessary to develop a way to find the equivalent standard deviation without using the sampling.

4.1 Shift of Probability of Failure

Equation (22) can be used at the optimum design obtained using the original random input and the original target probability of failure denoted by $P_{F_o}^{\mathrm{Tar}}$ as

$$P_F(\boldsymbol{\delta}) = \frac{P_{F_o}^{\mathrm{Tar}}}{\Phi(d_0)}\Phi\left(-\frac{1}{\sqrt{\sum_{i=1}^{N}(c_i\delta_i\sigma_i^o)^2}}\right) \tag{32}$$

since the current probability of failure at the optimum design should be the same as the target probability of failure. Then, the equivalent standard deviation can be found by setting $P_F(\boldsymbol{\delta}) = P_{F_{new}}^{\mathrm{Tar}}$. However, this approach apparently cannot be applied to the sampling-based RBDO since the objective of the proposed method is to efficiently find the same optimum design using the new target probability of failure and the equivalent standard deviation. Hence, Eq. (32) needs to be modified to find the equivalent standard deviation from the beginning of the design iterations.

If the new target probability of failure and equivalent standard deviation are used for the sampling-based RBDO, the probability of failure at the optimum design will be $P_{F_{new}}^{\mathrm{Tar}}$ and since $P_{F_{new}}^{\mathrm{Tar}} > P_{F_o}^{\mathrm{Tar}}$, we need a decreasing function of $\boldsymbol{\delta}$. Hence, using the inverse form of Eq. (22), the probability of failure at the optimum can be expressed as

$$P_F(\boldsymbol{\delta}) = \frac{P_{F_{new}}^{\mathrm{Tar}}\Phi(d_0)}{\Phi\left(-\frac{1}{\sqrt{\sum_{i=1}^{N}(c_i\delta_i\sigma_i^o)^2}}\right)} \tag{33}$$

and by letting $P_F(\boldsymbol{\delta})$ in Eq. (33) to be $P_{F_o}^{\mathrm{Tar}}$, $\boldsymbol{\delta}$ is obtained as

$$\sum_{i=1}^{N}(c_i\delta_i\sigma_i^o)^2 = \left(\frac{1}{\Phi^{-1}\left(\frac{\Phi(d_0)P_{F_{new}}^{\mathrm{Tar}}}{P_{F_o}^{\mathrm{Tar}}}\right)}\right)^2. \tag{34}$$

The solution of Eq. (34) cannot be uniquely obtained since there are N unknowns but only one equation. The easiest way of solving Eq. (34) is to assume all δ_i are the same as δ. However, there could be some cases where we cannot increase input standard deviations, for example, when input random variables cannot have negative values. In such cases, we can set up the ratio for those random variables as one and assume that the rest of the ratios are the same. After solving Eq. (34), the equivalent standard deviation for the i^{th} random variable is given by

$$\sigma_i^E = \delta_i\sigma_i^o. \tag{35}$$

However, as mentioned before, the current probability of failure at a design is not always $P_{F_{new}}^{\mathrm{Tar}}$ during the design optimization. Hence, the performance function needs to be shifted by α as

$$G^s(\mathbf{X}) = G(\mathbf{X}) + \alpha \tag{36}$$

such that the probability of failure at a current design is always $P_{F_{new}}^{\mathrm{Tar}}$. α can be easily obtained from function values at MCS samples. Using the shifted probability of failure, Eq. (34) can be used from the beginning and the coefficient d_0 is obtained by taking the partial derivative of Eq. (33) with respect to δ_j and using Eqs. (23) and (24) where the failure set Ω_F is defined as

$$\Omega_F \equiv \{\mathbf{x} : G(\mathbf{x}) + \alpha > 0\}. \tag{37}$$

As the design approaches the optimum design, the current probability of failure approaches the new target probability of failure and thus α converges to zero.

4.2 Algorithm

The sampling-based RBDO launches at the deterministic optimum design since it is usually closer to the RBDO optimum design than the initial design and accordingly the computational effort can be reduced. At the deterministic optimum design, the ratio $\boldsymbol{\delta}$ is first set up as 1 and the sampling is carried out for the reliability analysis. Using the reliability analysis result and Eq. (34), $\boldsymbol{\delta}$ is updated and the updated standard deviation is used as the input at the next design. For an RBDO problem with multiple constraints, even if δ is fixed as one parameter for multiple random variables, it could be different for different active constraints. In this case, the maximum $\boldsymbol{\delta}$ is selected to assure a reliable and safe optimum design. This could be another error source of the proposed method in addition to the probability of failure approximation shown in Eq. (18). This error will be studied in detail in Section 5.3 using a numerical example.

By using the sampling-based RBDO with the equivalent standard deviation, the new target probability of failure which is much larger than the original target probability of failure can be utilized resulting in the reduction of the number of samples used. Figure 2 shows the overall flowchart of the sampling-based RBDO with the equivalent standard deviation.

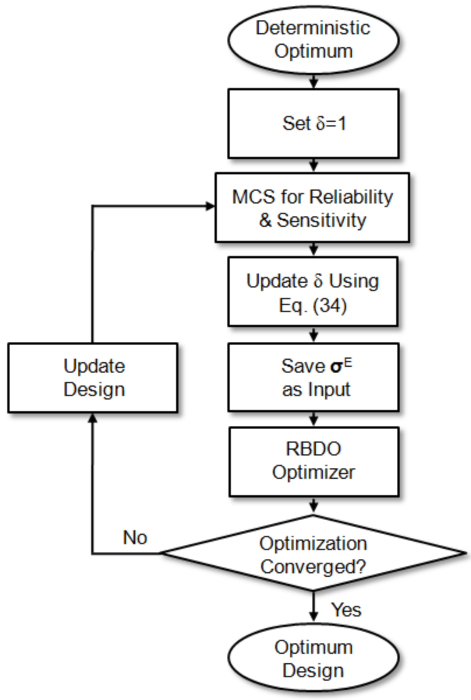


Figure 2. Flowchart of Sampling-based RBDO with Equivalent Standard Deviation

5. NUMERICAL EXAMPLES

Numerical studies are carried out in this section to verify the probability of failure estimation in terms of $\boldsymbol{\delta}$ proposed in Section 3 and the equivalent standard deviation for the sampling-based RBDO proposed in Section 4. Sections 5.1 and 5.2 show comparison studies between the MCS and proposed method for the probability of failure estimation using 2-D and 9-D mathematical examples, respectively. For the 2-D example, both independent and correlated random inputs are considered. Section 5.3 illustrates how the proposed equivalent standard deviation can be applied to solve a high-reliability RBDO model. For all tests, to concentrate on the proposed method and eliminate errors from surrogate models, true analytic functions are used instead of surrogate models, and the ratios (δ_i) in Eqs. (22) and (34) are assumed to be the same as δ for the simplicity of calculation.

5.1 Probability of Failure Estimation Using 2-D Mathematical Example

To verify how accurately the proposed probability of failure in terms of the input standard deviation can approximate the true one obtained by the MCS with 2 million samples, consider a 2-D highly nonlinear performance function [31] shown in Fig. 3 and expressed as

$$G(\mathbf{X}) = -1 + (Y-6)^2 + (Y-6)^3 - 0.6 \times (Y-6)^4 + Z \qquad (38)$$

where $\begin{Bmatrix} Y \\ Z \end{Bmatrix} = \begin{bmatrix} 0.9063 & 0.4226 \\ 0.4226 & -0.9063 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, X_1 and X_2 have $N(4.5, 0.3^2)$ and $N(2, 0.3^2)$, respectively, and they are statistically independent.

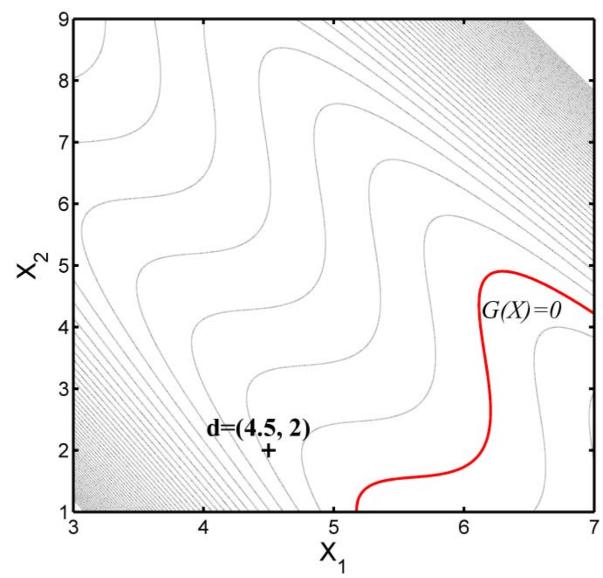


Figure 3. Shape of Highly Nonlinear Performance Function

Table 4 compares the probability of failure obtained using the MCS and the probability of failure estimated by the proposed method shown in Eq. (21), respectively, which is shown in Fig. 4, too. From the table and figure, we can see that the proposed probability of failure estimation works very well for a highly nonlinear performance function. Table 4 also shows that if the input standard deviation increases from 0.3 to 0.54, that is $\delta = 1.8$, the probability of failure increases more than 50 times. Inversely, if we want the current probability of failure to increase by 50 times, then we can find the corresponding standard deviation, which is used to find the equivalent standard deviation. If the increased probability of failure and equivalent standard deviation are used for the sampling-based RBDO, the total number of MCS samples will reduce to less than 2% of the number of MCS samples required to obtain the same accuracy using the original standard deviation.

Table 4. Comparison of Probability of Failure

		$\delta=1.0$	$\delta=1.2$	$\delta=1.4$	$\delta=1.6$	$\delta=1.8$
P_F, %	Estimated	0.0299	0.1663	0.4821	0.9833	1.6284
	MCS	0.0299	0.1678	0.4881	0.9912	1.6483
Error, %		0.00	0.89	1.25	0.80	1.21

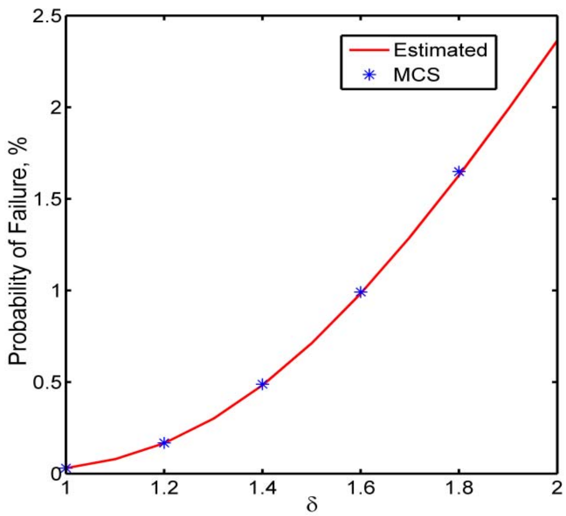


Figure 4. Comparison of MCS and Estimated Probability of Failure for Independent Case

To test the probability of failure estimation for correlated random input, suppose that X_1 and X_2 have $N(4.5, 0.45^2)$ and $N(2, 0.45^2)$, and they are correlated with the Clayton copula (τ=0.5). The same performance function in Eq. (38) is still used for the test. Table 5 and Fig. 5 show the result of the comparison test. In this case, due to the correlation effect, error between the MCS and the estimated probability of failure becomes relatively larger than the independent case. However, it is still accurate enough up to $\delta = 1.4$, which means the probability of failure can be increased by about 20 times and 5% of the number of MCS samples will be required for the same accuracy for the sampling-based RBDO.

Table 5. Comparison of Probability of Failure

		$\delta=1.0$	$\delta=1.2$	$\delta=1.4$	$\delta=1.6$	$\delta=1.8$
P_F, %	Estimated	0.0298	0.1823	0.5609	1.1891	2.0229
	MCS	0.0298	0.1894	0.5792	1.3124	2.3720
Error, %		0.00	3.75	3.16	9.39	14.72

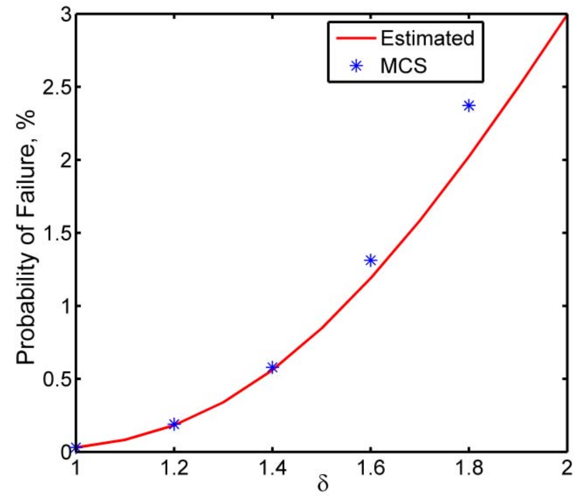


Figure 5. Comparison of MCS and Estimated Probability of Failure for Correlated Case

5.2 Probability of Failure Estimation Using 9-D Mathematical Example

To verify whether the proposed method works for high-dimensional problems, consider a 9-D polynomial function, which is known as the extended Rosenbrock function [32] and modified for the purpose of the probability of failure calculation,

$$G(\mathbf{X}) = \sum_{i=1}^{8} \left[(1 - X_i)^2 + 100(X_{i+1} - X_i^2)^2 \right] - 1136, \quad -5 \le X_i \le 10 \text{ for } i = 1, \cdots, 9 \tag{39}$$

where the properties of nine random variables are shown in Table 6. For this problem, all random variables are assumed to be statistically independent.

Table 6. Properties of Random Variables

Random Variables	Distribution	Mean	Standard Deviation
X_1~X_9	Normal	1.0	0.2

Table 7 and Fig. 6 show the result of the comparison test. For high-dimensional problems, the probability of failure is very sensitive to the change of the input standard deviations as shown in Table 7. Hence, in this test, the increment of standard deviation is tested only up to $\delta = 1.4$. Table 7 shows that by increasing the input standard deviation by 1.4 times the probability of failure increases almost 100 times with 11% error. This implies that if the increased standard deviation is used for the sampling-based RBDO, the total number of MCS samples reduces to less than 1% of the number of MCS samples required to obtain the same accuracy using the original standard deviation. This significant reduction of the number of MCS samples used will be shown in the next section.

Table 7. Comparison of Probability of Failure

		$\delta=1.0$	$\delta=1.1$	$\delta=1.2$	$\delta=1.3$	$\delta=1.4$
P_F, %	Estimated	0.0088	0.0422	0.1408	0.3630	0.7759

MCS	0.0088	0.0402	0.1488	0.3992	0.8718
Error, %	0.00	4.97	5.37	9.07	11.00

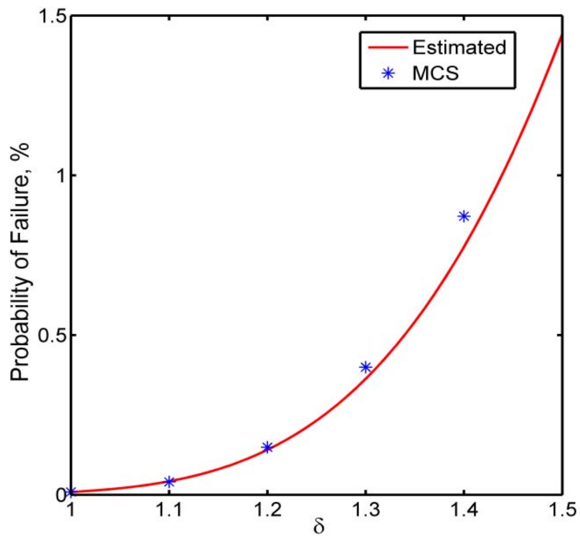


Figure 6. Comparison of MCS and Estimated Probability of Failure for High Dimensional Case

5.3 Sampling-Based RBDO with Equivalent Standard Deviation

To see how the proposed equivalent standard deviation can reduce the computational effort of the sampling-based RBDO, consider a 2-D mathematical RBDO problem, which is formulated to

$$\begin{aligned} &\text{minimize} \quad C(\mathbf{d}) = -\frac{(d_1+d_2-10)^2}{30} - \frac{(d_1-d_2+10)^2}{120} \\ &\text{subject to} \quad P(G_j(\mathbf{X}(\mathbf{d}))>0) \le P_{F_j}^{\text{Tar}}, \quad j=1\sim 3 \qquad (40) \\ &\qquad \mathbf{d}^L \le \mathbf{d} \le \mathbf{d}^U, \quad \mathbf{d}\in \mathrm{R}^2 \text{ and } \mathbf{X}\in \mathrm{R}^2 \end{aligned}$$

where three constraints are given by

$$\begin{aligned} G_1(\mathbf{X}) &= 1-\frac{X_1^2 X_2}{20} \\ G_2(\mathbf{X}) &= -1+(Y-6)^2+(Y-6)^3-0.6\times(Y-6)^4+Z \qquad (41) \\ G_3(\mathbf{X}) &= 1-\frac{80}{X_1^2+8X_2+5} \end{aligned}$$

where $\begin{Bmatrix} Y \\ Z \end{Bmatrix} = \begin{bmatrix} 0.9063 & 0.4226 \\ 0.4226 & -0.9063 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, and are drawn in Fig. 7. The properties of two random variables are shown in Table 8, and they are assumed to be independent. In Eq. (40), the original target probability of failure ($P_{F_o}^{\text{Tar}}$) is set up as 0.003167% for all three constraints, which is a 4σ design.

Table 8. Properties of Random Variables

Random Variables	Distribution	d^L	d^O	d^U	Standard Deviation
X_1	Normal	0.0	5.0	10.0	0.2
X_2	Normal	0.0	5.0	10.0	0.2

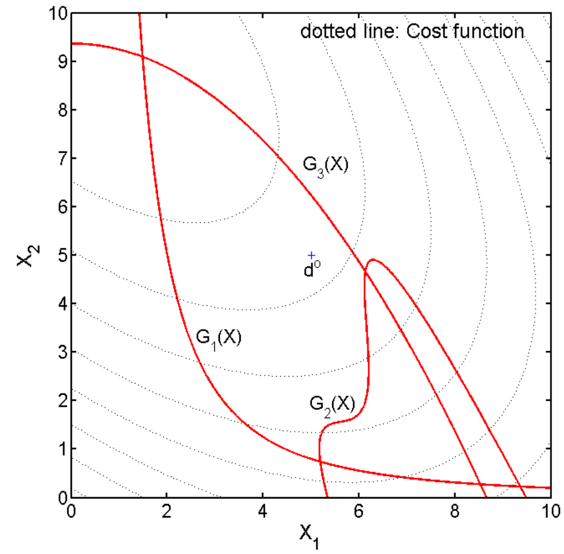


Figure 7. Shape of Constraint and Cost Functions

Using Eq. (9), the number of the MCS samples to accurately estimate the target probability of failure is 50 million assuming $\varepsilon_{MCS}=5\%$, which will make the sampling-based RBDO slow and even slower when combined with implicit surrogate models such as the Kriging model. To carry out the sampling-based RBDO, the deterministic optimum is first found at $\mathbf{d}_{\text{dopt}}$=(5.1956, 0.7407) where the sampling-based RBDO with the original random input and target probability of failure is launched and the RBDO optimum is found at $\mathbf{d}_{\text{ropt}}$=(4.6184, 1.8247). To find the RBDO optimum, 25 million MCS samples are used when constraints are not active and 50 million MCS samples are used when active. Detailed information on how to carry out the sampling-based RBDO is shown in Ref. 22.

For the test of the equivalent standard deviation, 5 different new target probabilities of failure which are obtained by multiplying the original target probability of failure by 10,20,30,40, and 50, respectively, are considered as listed in Table 9. The third column of Table 9 shows the number of the MCS samples required to obtain the probability of failure with the same accuracy ($\varepsilon_{MCS}=5\%$) as the original random input. From the third column of Table 9, it can be easily shown how drastically the number of the MCS samples is reduced from 50 million. With 5 different new target probabilities of failure, 5 different sampling-based RBDOs are carried out with the MCS samples shown in Table 9.

Table 9. New Target Probability of Failure

Case	$P_{F_{new}}^{\text{Tar}}$,%	No. of MCS Samples
Case 1	0.03167	5.05 M
Case 2	0.06334	2.52 M
Case 3	0.09501	1.68 M
Case 4	0.12668	1.26 M
Case 5	0.15835	1.01 M

Table 10 compares the sampling-based RBDO results of each case with the one obtained from the original input and target probability of failure. At each optimum design, the MCS with 50 million samples is carried out to check the accuracy of the sampling-based RBDO with the equivalent standard deviation. Table 10 shows that the maximum error in terms of the probability of failure estimation becomes larger as the new target probability of failure become larger. This is mainly because the difference between two δs for two active constraints is larger as the new target probability of failure become larger, and only one δ is used for the design optimization. Another error source is the high nonlinearity of the second constraint function as shown in Fig. 7. Even with relatively large error of the probability of failure calculation, the optimum designs are very close to the original result.

Table 10. Comparison of Five RBDO Cases

Case	Optimum Design	MCS		Max Error,%
		P_{F_1},%	P_{F_2},%	
Case 1	4.6206, 1.8258	0.002964	0.003256	6.4
Case 2	4.6215, 1.8242	0.003120	0.003468	9.5
Case 3	4.6235, 1.8263	0.002850	0.003396	10.0
Case 4	4.6259, 1.8271	0.002768	0.003564	12.5
Case 5	4.6286, 1.8280	0.002660	0.003730	17.8
Original	4.6184, 1.8247	0.003132	0.003218	1.6

The proposed sampling-based RBDO with the equivalent standard deviation is originally developed to be combined with surrogate models to enhance the computational efficiency. However, it can be applicable to the sampling-based RBDO using actual computer simulations if the model is not computationally demanding since the proposed method can reduce the number of samples significantly, especially when used in conjunction with more efficient sampling schemes than the MCS. In that case, error from surrogate models is eliminated, which makes the proposed sampling-based RBDO more accurate.

6. CONCLUSIONS

To enhance the computational efficiency of the sampling-based RBDO, a methodology to convert an RBDO problem with very small probability of failure to an RBDO problem with relatively large probability of failure by increasing input standard deviations is proposed, which can be used in conjunction with surrogate models and improved sampling schemes. The first-order score function for the input standard deviation is used to derive the probability of failure in terms of the input standard deviation for both independent and correlated random inputs. The derived probability of failure is then used to update the target probability of failure and find the equivalent standard deviation to be used for the sampling-based RBDO. Numerical examples show the accuracy of the proposed probability of failure in terms of input standard deviations and demonstrate that the sampling-based RBDO with the equivalent standard deviation yields a similar optimum design obtained using the original random input with significantly enhanced computational efforts. To further improve the accuracy of the proposed method, a new equivalent standard deviation concept is being investigated and the new concept will be tested using large-scale real engineering applications in future study.

7. ACKNOWLEDGEMENT

Research is jointly supported by the Automotive Research Center, which is sponsored by the U.S. Army TARDEC and ARO Project W911NF-09-1-0250. These supports are greatly appreciated.

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